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PROOF OF A CONJECTURE OF BOLLOBÁS AND ELDRIDGE FOR GRAPHS OF MAXIMUM DEGREE THREE*

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Let G_1 and G_2 be simple graphs on n vertices. If there are edge-disjoint copies of G_1 and G_2 in K_n, then we say there is a packing of G_1 and G_2. A conjecture of Bollobás and Eldridge [5] asserts that if $(\Delta(G_1)+1)(\Delta(G_2)+1)\le n+1$ then there is a packing of G_1 and G_2. We prove this conjecture when $\Delta(G_1)=3$, for sufficiently large n.

1. Introduction

In this paper we will only consider graphs without loops and multiple edges. $\delta(G)$ and $\Delta(G)$ are the minimum degree and maximum degree of the graph G (if it is clear from the context G might be omitted).

An important question in extremal graph theory is the sufficient condition on the minimum degree of a graph which guarantees the existence of a spanning subgraph. In [5] Bollobás and Eldridge conjectured that if $(\Delta(G_1)+1)(\Delta(G_2)+1)\le n+1$, then there will be a packing of G_1 and G_2. Their conjecture can be restated in the following complementary form:

Conjecture 1 (Bollobás–Eldridge). If G is a graph on n vertices with

$$\delta(G)\ge \frac{kn-1}{k+1}$$

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then G contains any n vertex graph H with $\Delta(H)=k$.

Certain special cases of this conjecture have been proved by Corrádi and Hajnal [6], Hajnal and Szemerédi [7], Aigner and Brandt [3] and Alon and Fischer [4]. The goal of this paper is to show that the conjecture is true for all sufficiently large n when $k=3$:

Theorem 2. *There exists an n_0 such that for all $n \geq n_0$ the following statement holds: Let H be a graph on n vertices and $\Delta(H)=3$. If G is any n vertex graph such that*

$$\delta(G) \geq \frac{3n-1}{4}$$

then it contains H as a spanning subgraph.

In proving Theorem 2, we need to distinguish between two cases depending on the structure of H. We call H *extremal*, if except a small proportion of the vertices, it is a union of disjoint K_4s. The proof for this case is different from the proof of the *non-extremal* case.

2. Notation and Definitions

For a graph G, $V(G)$ and $E(G)$ will denote its vertex-set and edge-set, respectively. For any vertex v, $\deg_G(v)$ is the degree of the vertex v, $\deg_G(v,X)$ is the number of the neighbors of v in X, and $e(X,Y)$ is the number of the edges between X and Y. $N_G(v)$ is the set of neighbors of v and $N_G(v,X)$ is the set of neighbors of v in X.

A bipartite graph G with color-classes A and B and edge-set E will be denoted by $G=(A,B,E)$. The *density* between disjoint sets X and Y is defined as:

$$d(X,Y)=\frac{e(X,Y)}{|X||Y|}.$$

In the proof of Theorem 2, Szemerédi's Regularity Lemma [10], [9] plays a pivotal role. We will need the following definition to state the Regularity Lemma.

Definition 1 (Regularity condition). Let $\varepsilon>0$. A pair (A,B) of disjoint vertex-sets in G is ε-regular if for every $X \subset A$ and $Y \subset B$, satisfying

$$|X|>\varepsilon|A|,\ |Y|>\varepsilon|B|$$

we have

$$|d(X,Y)-d(A,B)|<\varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; namely, the density of any reasonably large subgraph is almost the same as the density of the regular pair.

We will use the following form of the Regularity Lemma:

Lemma 3 (Degree Form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G=(V,E)$ is any graph and $d\in[0,1]$ is any real number, then there is a partition of the vertex set V into $\ell+1$ clusters $V_0, V_1, \dots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \le M$,
- $|V_0| \le \varepsilon |V|$,
- *all clusters V_i, $i\ge 1$, are of the same size* $m\left(\le \lfloor \frac{|V|}{\ell} \rfloor < \varepsilon|V|\right)$,
- $\deg_{G'}(v) > \deg_G(v) - (d+\varepsilon)|V|$ *for all* $v \in V$,
- $G'|_{V_i} = \emptyset$ *(V_i is an independent set in G') for all $i \ge 1$,*
- *all pairs (V_i, V_j), $1 \le i < j \le \ell$, are ε-regular, each with density either 0 or greater than d in G'.*

Often we call V_0 the *exceptional cluster*. A stronger one-sided property of regular pairs is super-regularity:

Definition 2 (Super-Regularity condition). Given a graph G and two disjoint subsets of its vertices A and B, the pair (A,B) is (ε,d)-super-regular, if it is ε-regular and furthermore,

$$\deg(a) > d|B|, \text{ for all } a \in A,$$

and

$$\deg(b) > d|A|, \text{ for all } b \in B.$$

Throughout the paper we will apply the relation "$\ll$": $a \ll b$, if a is sufficiently smaller, than b.

3. The structure of G

We will apply Lemma 3 to the graph G of Theorem 2 with parameters $0 < \varepsilon \ll d \ll 1$, creating a partition of $V(G)$ into clusters $V_0, V_1, \dots, V_\ell$. The edge distribution among non-exceptional clusters of G' is reflected in the following auxiliary graph G_r. The vertex set of the *reduced graph* G_r is the set of clusters $\{V_1, \dots, V_\ell\}$. There is an edge between two vertices in G_r if the corresponding clusters form an ε-regular pair in G', with density exceeding d.

Since $\delta(G') \geq (3/4-(d+\varepsilon))n$, it follows that for $\eta = 10d$

$$\delta(G_r) \geq \left(\frac{3}{4} - \eta\right)\ell.$$

Next, we will add a few extra vertices to G_r and connect them to all the original vertices of G_r, so their minimum degree goes up to 3/4 times the number of vertices. Using the theorem of Hajnal and Szemerédi [7] the resulting graph can be tiled with disjoint copies of cliques of size four. Returning to our original G_r, apart from $12\eta\ell$, all the clusters in G_r can be covered by disjoint K_4s. We will also add the vertices contained in these $12\eta\ell$ clusters to the exceptional cluster V_0. For simplicity, we let G_r and V_0 denote the remaining clusters and resulting exceptional vertices, respectively. Using θ to denote 13η, it is easy to see that $|V_0| \leq \theta$ and so the following structural theorem holds:

Theorem 4 (First Decomposition Theorem). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any n–graph satisfying $\delta(G) \geq \frac{3n-1}{4}$ and $d \in [0,1]$ is any real number, then there is a partition of the vertex set V into $\ell+1$ clusters $V_0, V_1, \ldots, V_\ell$, and a corresponding reduced graph G_r with vertex set $\{1, 2, \ldots, \ell\}$ with a K_4 cover of its vertices. Furthermore, there is a subgraph G' of G with the following properties:*

- $\ell \leq M$,
- $|V_0| \leq 130d|V| = \theta|V|$,
- *all clusters V_i, $i \geq 1$, are of the same size* $m \left(\leq \lfloor \frac{|V|}{\ell} \rfloor < \varepsilon|V|\right)$,
- $G'|_{V_i} = \emptyset$ *(V_i is an independent set in G') for all $i \geq 1$,*
- $\delta(G_r) \geq (\frac{3}{4} - \theta)\ell$,
- *all pairs (V_i, V_j), $1 \leq i < j \leq \ell$, $ij \in E(G_r)$, are ε-regular, each has density greater than d in G'.*

In the proof of Theorem 2 for an extremal H we will use a decomposition theorem due to Abbasi [1,2], which is more appropriate for certain cases of a graph H and produces more insight to the structure of the graph G. To begin with, we need the following definition:

Definition 3. A set $I \subset V$ of size p is called α-quasi-independent if I contains at most αp^2 edges.

Theorem 5 (Second Decomposition Theorem). *Let $0 < \varepsilon \ll d \ll \rho$, and let k, h be two positive integers. Then there exists an integer ℓ and a threshold n_0 such that if $G = (V, E)$ is an arbitrary n-vertex graph with $\delta(G) \geq \frac{(k-1)n-1}{k}$, and $n \geq n_0$, then one of the following two structures can be*

recognized in G:
(i) G contains a $(2k^3\rho)$-quasi-independent set of size $\frac{n}{k}$;
(ii) we can partition V into ℓ clusters $V_1,\dots,V_\ell$ and decompose the clusters into blocks $\mathcal{A}_1,\dots,\mathcal{A}_s$, $\mathcal{B}_1,\dots,\mathcal{B}_r$ such that the following properties hold:

- **Decomposition into k and $(k+1)$-blocks.** *Each $\mathcal{A}$-block contains k clusters, and each $\mathcal{B}$-block contains $k+1$ clusters. We will refer to an $\mathcal{A}_i$ as a k-block, and a $\mathcal{B}_j$ as a $(k+1)$-block.*
- **Many $(k+1)$-blocks.** *The number of $(k+1)$-blocks is at least $\rho\ell$.*
- **Balance in the blocks.**

$$\frac{n}{\ell}(1-\rho) \le |V_i| \le \frac{n}{\ell}(1+\rho).$$

- **Perfect balance in the k-blocks.** *For every k-block $\mathcal{A}_i$ each cluster contains m_i vertices, and h divides m_i.*
- **Super-regularity within blocks.** *Any two clusters V_a and V_b in a k-block form an (ε,d)-super-regular pair. Similarly, any two clusters V_a and V_b in a $(k+1)$-block form an (ε,d)-super-regular pair.*
- **Connectivity from k-blocks to $(k+1)$-blocks.** *For every k-block $\mathcal{A}_i = \{V_1,\dots,V_k\}$ there is a $(k+1)$-block $\mathcal{B}_j = \{W_0,\dots,W_k\}$ (with its clusters ordered appropriately) such that (V_p, W_q) is (ε,d)-regular for all $p \neq q$.*
- **Connectivity among $(k+1)$-blocks.** *Let us construct an auxiliary graph F on r vertices $\{b_1,\dots,b_r\}$, every vertex corresponding to a $(k+1)$-block. We connect b_i and b_j if there is a cluster $W \in \mathcal{B}_i = \{W_0,\dots,W_k\}$ such that there are at least $k-1$ (ε,d)-regular edges connecting W with the clusters of the $(k+1)$-block $\mathcal{B}_j$. Then F is a connected graph.*

If in the above theorem part (i) holds, we will refer to G as an *extremal graph*, otherwise we will consider G *non-extremal*.

In short, we can either cover the clusters of the reduced graph G_r by a non-negligible number of disjoint $(k+1)$-cliques and k-cliques, or the original graph G is an extremal graph, *i.e.*, it contains a large quasi-independent set. Moreover, if G is not extremal, there is a well-defined connectivity structure among the cliques covering G_r. In our application of the Second Decomposition Theorem, we need some insight to its proof. It starts by applying the Degree Form of the Regularity Lemma, and finds the disjoint K_{k+1}-K_k cover of the clusters with enough K_{k+1}s (if G is non-extremal). During this process the original clusters can be cut into at most k equal pieces. Observe, that this step will not violate the regularity condition on (ε,d)-regular edges; the only change is that ε will be replaced by $k\varepsilon$. Next, the edges participating in the clique-cover will be converted to super-regular pairs, and as a result some extra vertices from each cluster will be added to the exceptional cluster V_0. Our final task will be to distribute the vertices of V_0 among the cliques so that the super-regularity of the edges is preserved. Since in our covering there are enough K_{k+1}s, by the minimum degree condition either a vertex has big degrees to all the k clusters of some k-clique, or it should have big degrees to at least k clusters of some $(k+1)$-cliques. In the former case the exceptional vertex can be added to any of the clusters in the k-clique. In the latter case, the vertex will be added to a cluster in which it has the smallest number of neighbors. Consequently, it cannot be guaranteed that all the clusters in the $(k+1)$-cliques are perfectly balanced. Finally, to guarantee that all the clusters of the k-cliques are balanced, we can use the connectivity structure among the k and $(k+1)$-cliques.

4. An outline of the proof

Without loss of generality, we might assume that the graph H is an "almost 3-regular graph" (with at most three vertices having degree less than 3). In the proof of Theorem 2, depending on the structure of H we will consider three different cases. Prior to describing these cases we need to identify whether H is an extremal graph:

Definition 4. Let $c=c(\varepsilon)$ be a constant depending on ε (to be determined later). The graph H is said to be non-extremal if at least cn vertices do not belong to 4-cliques. If H is almost the union of 4-cliques, it is said to be an *extremal* graph.

To prove Theorem 2 for a non-extremal H we will utilize the First Decomposition Theorem in conjunction with the following result (see Sections 5.1–5.3 for the proof):

Lemma 6 (Main Lemma). *If H is non-extremal, we can partition its vertex set into clusters $L_0, L_1, L_2, \dots, L_\ell$ and determine a bijective mapping φ between L_0 and V_0 in such a way that*

- $|L_i| = |V_i|$ $(i=1,2,\dots,\ell)$,
- *the elements of L_0 are at least of distance 4 from each other,*
- *L_is $(i=1,\dots,\ell)$ are independent in H,*
- $|N(L_0)\cap L_i| \le \frac{100|L_i|}{\ell}$ *for* $i=1,2,\dots,\ell$,
- *if $(x,y)\in E(H)$ and $x\in L_i$, $y\in L_j$ $(i,j>0)$ then $(V_i,V_j)\in E(G_r)$,*
- *if $(x,y)\in E(H)$ and $x\in L_0$, $y\in L_j$ $(j>0)$ then* $\deg(\varphi(x),V_j)\ge \frac{|V_j|}{10} = \frac{m}{10}$.

Once φ is in hand, we have an initial assignment of H to the clusters of G_r and an exceptional-cluster V_0. To finish the proof of Theorem 2, we need to find a one-to-one assignment between H and G vertices. This last step of the proof will follow a similar line of argument as the Blow-up Lemma [8] (see Sections 5.4–5.7).

If the graph H is extremal (disjoint union of K_4s and perhaps a small residue $\widetilde{H}$), we will consider two separate cases depending on whether G is extremal or non-extremal. In the latter case, using a procedure similar to the case of a non-extremal H, we will first embed $\widetilde{H}$ into G. Then the leftover of H (which is the union of K_4s) will be embedded to the free vertices in G. After embedding $\widetilde{H}$ into G, the minimum degree condition is not true for the leftover of G, consequently, we cannot apply the Hajnal–Szemerédi Theorem. An alternative way for this case will be presented in Section 6. Finally, when both H and G are extremal, the presence of a large quasi-independent set in G will give us a lot of room for embedding H using a "brute force" approach (see Section 7).

5. *H* is non-extremal

Let I denote a maximum independent set in graph H. Clearly, $|I| \geq \frac{n}{4}$. The subgraph H' induced by $V(H)-I$ is a graph with maximum degree 2, i.e., its components are cycles and paths. By removing a subset Q of size at most $\sqrt{n}$ from H', we can guarantee that the remaining components in H' are all short (not longer than $\sqrt{n}$). We will extend the set Q by including those I vertices with a neighbor in Q. For simplicity we will use Q to denote this newly formed set of vertices. Note that $|Q| \leq 4\sqrt{n}$. Next, the vertices in $I-Q$ will be further partitioned to $\{I_1, I_2, I_3\}$ based on the following conditions:

- $x \in I_1$ iff the neighbors of x in H' belong to different components.
- $y \in I_2$ iff the neighbors of y in H' belong to exactly two different components,
  or it has only two neighbors, in the same component.
- $z \in I_3$ iff all three neighbors of z in H' belong to the same component.

The following characterization of the components in $H'-Q$ is a direct consequence of I being a maximum independent set.

Lemma 7. *Every I_3-vertex is connected to either a triangle or a path in H'. In the former case, there can be only one such I_3-vertex connected to that triangle.* ∎

For reasons which will be clear in Section 5, those I_3-vertices which are connected to a path in H' will be also added to I_2. In terms of embedding of H in G, such vertices have similar properties as the ones in I_2.

Let I' be a maximum subset of vertices in I, which are far from each other, i.e., any two points in I' are of distance at least 4. Since H has maximum degree 3, it is easy to see that $|I'| \geq \frac{1}{18}|I|$. We let I'_1, I'_2 and I'_3 denote $I_1 \cap I'$, $I_2 \cap I'$ and $I_3 \cap I'$, respectively. The embedding of a non-extremal H will be performed in the following two steps:

I. Assign the vertices of $H-Q$ to clusters in G_r such that if $x, y \in V(H)$ satisfy $(x,y) \in E(H)$, and they are mapped to V_x and V_y in G_r, then $(V_x, V_y) \in E(G_r)$. There should be $\frac{n}{\ell} + o(n)$ vertices of H mapped to every cluster in G_r. We will construct this mapping in several steps, by first assigning the vertices of H' (see Section 5.1) and then those in $I-Q$ to clusters of G_r (see Section 5.2).

II. Let L_i denote those H vertices that have been assigned to the cluster V_i (we will also use $L(V_i)$ to denote the same set), for all $1 \leq i \leq \ell$. At the end of the previous step, none of the H-vertices have been assigned to the exceptional cluster V_0 and consequently $|L_i| \geq |V_i|$ for all $1 \leq i \leq \ell$ (we may assume that $|V_0| = \theta n$, since otherwise we can choose at most θm vertices from each cluster in G_r to complete V_0). By appropriately removing $|L_i| - |V_i|$ vertices from each L_i, we form a set L_0. We will also add the vertices in Q to L_0. Consequently, L_0 will have the same size as V_0. To map V_0 to H-vertices, we will perform a preprocessing step (Section 5.3) during which we might switch some of the vertices in L_0 and other L_i-vertices. Nevertheless, the size of L_0 and the vertices in V_0 will stay the same. Finally, L_0 and $H - L_0$-vertices will be embedded into V_0 and $V(G) - V_0$, respectively (see Sections 5.4–5.7).

5.1. Mapping H' to clusters of G_r

We start by assigning the vertices of $H' - Q$ to clusters of G_r. For every component $C \in H' - Q$, randomly and independently choose a triangle T from the K_4-cover of G_r. If C is 2-colorable, choose a random edge (V_i, V_j) of T, and distribute the vertices of C between the two sets $L(V_i)$ and $L(V_j)$ randomly. Otherwise, if C is an odd cycle, distribute its vertices among the clusters of T randomly. In both cases adjacent vertices in C should not be mapped to the same cluster in T. This procedure will guarantee an almost even distribution of the vertices of $H' - Q$ into the clusters of G_r:

Lemma 8. *Let* $L(V_i)$, $1\le i\le \ell$ *denote the set of vertices assigned to* V_i *after distributing the vertices of* $H'-Q$ *using the above procedure. Then, with high probability* $|L(V_i)|=\frac{n'}{\ell}\pm o(n)$, *with* $n'=|H'-Q|$.

Proof. Applying Chebyshev's inequality gives the proof of the lemma. ∎

Next, we will argue that an appropriate distribution of $H'-Q$ among the clusters of G_r will facilitate an even assignment of the vertices of I to the clusters of G_r. Let V be a cluster in G_r, we define the *associated list* $\mathcal{A}(V)$ as $\{y: y\in I, x\in L(V), (x,y)\in E(H)\}$, which is the set of independent vertices of H with a neighbor assigned to the cluster V. Given the associated lists $\mathcal{A}(V_i), \mathcal{A}(V_j), \mathcal{A}(V_k)$, for each $1\le p\le 3$, the random variables $R'_p(V_i,V_j,V_k)=|I'_p\cap\mathcal{A}(V_i)\cap\mathcal{A}(V_j)\cap\mathcal{A}(V_k)|$ and $R_p(V_i,V_j,V_k)=|(I_p-I'_p)\cap\mathcal{A}(V_i)\cap\mathcal{A}(V_j)\cap\mathcal{A}(V_k)|$ will denote the number of independent vertices in I'_p and $I_p-I'_p$, respectively, which have neighbors assigned to the clusters V_i, V_j and V_k.

Lemma 9. *For any three clusters* V_i, V_j, *and* V_k *in* G_r *the following inequalities hold:*

$$\Pr\left[|R'_1(V_i,V_j,V_k)-\mathrm{E}[R'_1(V_i,V_j,V_k)]|>n^{\frac{4}{5}}\right]=o(1),$$

$$\Pr\left[|R_1(V_i,V_j,V_k)-\mathrm{E}[R_1(V_i,V_j,V_k)]|>n^{\frac{4}{5}}\right]=o(1).$$

Moreover, for all $1\le i\le \ell$ *if* (V_j,V_k) *is an edge in the* K_4*-cover of* G_r *then*

$$\Pr\left[|R'_2(V_i,V_j,V_k)-\mathrm{E}[R'_2(V_i,V_j,V_k)]|>n^{\frac{4}{5}}\right]=o(1),$$

$$\Pr\left[|R_2(V_i,V_j,V_k)-\mathrm{E}[R_2(V_i,V_j,V_k)]|>n^{\frac{4}{5}}\right]=o(1).$$

Finally, if V_i, V_j, *and* V_k *are the clusters of a triangle in the* K_4*-cover of* G_r *then*

$$\Pr\left[|R'_3(V_i,V_j,V_k)-\mathrm{E}[R'_3(V_i,V_j,V_k)]|>n^{\frac{4}{5}}\right]=o(1),$$

$$\Pr\left[|R_3(V_i,V_j,V_k)-\mathrm{E}[R_3(V_i,V_j,V_k)]|>n^{\frac{4}{5}}\right]=o(1).$$

Proof. Similar to the proof of Lemma 8, again we omit the details. ∎

We need the following simple corollary of the above lemmas.

Lemma 10. *For any two cluster triplets* (V_i,V_j,V_k) *and* (V'_i,V'_j,V'_k) *in* G_r *the following inequalities hold:*

$$\Pr\left[|R'_1(V_i,V_j,V_k)-R'_1(V'_i,V'_j,V'_k)|>n^{\frac{4}{5}}\right]=o(1),$$

$$\Pr\left[|R_1(V_i,V_j,V_k)-R_1(V'_i,V'_j,V'_k)|>n^{\frac{4}{5}}\right]=o(1).$$

Moreover, for all $1 \le i, i' \le \ell$ *if* (V_j, V_k) *and* (V_j', V_k') *are two edges in the* K_4*-cover of* G_r *then*

$$\Pr\left[|R_2'(V_i,V_j,V_k) - R_2'(V_i',V_j',V_k')| > n^{\frac{4}{5}}\right] = o(1),$$

$$\Pr\left[|R_2(V_i,V_j,V_k) - R_2(V_i',V_j',V_k')| > n^{\frac{4}{5}}\right] = o(1).$$

Finally, if V_i, V_j, V_k *and* V_i', V_j', V_k' *are triangles in the* K_4*-cover of* G_r *then*

$$\Pr\left[|R_3'(V_i,V_j,V_k) - R_3'(V_i',V_j',V_k')| > n^{\frac{4}{5}}\right] = o(1),$$

$$\Pr\left[|R_3(V_i,V_j,V_k) - R_3(V_i',V_j',V_k')| > n^{\frac{4}{5}}\right] = o(1).$$

In short, KULemma10 states that for a proper distribution of the vertices in $H' - Q$, the mapped vertices to any three clusters in G_r will have almost the same number of neighbors in any of the classes I_1', I_2', I_3', and their complements with respect to I.

5.2. Assigning $I - Q$ to clusters of G_r

In this section we will present a consistent assignment of the vertices in $I - Q$ to the clusters of G_r, where consistency means that for any pair of vertices x, y in the graph H assigned to clusters V_i and V_j, respectively, if $(x,y) \in E(H)$ then $(V_i, V_j) \in E(G_r)$. As we will see, such assignments can be formulated as special matching problems. (In order to finish the embedding of H into G, some of the H-vertices should be assigned to the exceptional cluster V_0, which will be done through a *preprocessing* procedure, discussed later.) For a bipartite graph $A = (V, T, E(A))$ where $|T| = q|V|$ for some positive integer q, $M \subset E(A)$ is a *proportional matching* if every $v \in V$ is adjacent to exactly q vertices in T and every $u \in T$ is adjacent to exactly one V vertex in M. In order to show that A contains a proportional matching we will check the König–Hall conditions, that is, for every subset U of V, its neighborhood in T should satisfy $|N_A(U,T)| \ge q|U|$. One can easily see this from the construction of an auxiliary graph: substitute every $v \in V$ with q instances $v_1, \dots, v_q$, and if (v,u) $(u \in T)$ was an edge, then connect the v_is to u for all $1 \le i \le q$. This auxiliary graph has a perfect matching if and only if A has a proportional matching.

Besides this kind of matching we are going to need another kind of matching about which we demand that the “loads of the vertices” are distributed more evenly. We say A *allows a strong proportional matching with respect to* μ $(0 < \mu \ll 1)$ if there is a proportional matching in the following bipartite graph A'. Its color classes are V and T'. For every vertex $u\in T$, we add $\frac{\ell}{\mu}$ copies, $u_1,\dots,u_{\frac{\ell}{\mu}}$, to T'. If $N_A(u)=\{v_1,\dots,v_s\}$ then we will have the following edges: (u_i,v_i) for $1\le i\le s$, and (u_j,v_i) where $1\le i\le s$ and $s<j\le\frac{\ell}{\mu}$. In other words, the first s copies of u have degree 1, while the others have the same degree, s. The existence of a strong proportional matching can be checked through the *strong König–Hall conditions*: one can see that for $U\subset V$ $|N_A(U)|(1-\mu)\le|N_{A'}(U)|$. Using this fact we can prove the existence of a strong proportional matching, and at the same time the existence of a proportional matching as well. We will see that both these matchings are needed to assign the vertices of I and Q to clusters of G_r.

Let us start by constructing our first bipartite graph A. For the reduced graph G_r we define $A=A(G_r)$ in the following way: $A=A(V,T,E(A))$, with $V=V(G_r)$, $T=\{\{v_i,v_j,v_k\}: v_i,v_j,v_k\in V(G_r), v_i,v_j,v_k$ all distinct$\}$, and $E(A)=\{(u,\{v_i,v_j,v_k\}):(u,v_i),(u,v_j),(u,v_k)\in E(G_r)\}$. Let $0<\theta\ll\mu\ll 1$.

Lemma 11. *There is a proportional matching in the bipartite graph A, and it allows a strong proportional matching with respect to μ.*

Proof. We will check the "strong" König–Hall conditions for all possible size subsets of V.

- For any $v\in V$ we have
$$|N_A(v,T)|\ge\left(\frac{3}{4}-\theta\right)^3|T|.$$
- Let $U\subset V$ with $|U|=(\frac{3}{4}-\theta)^3(1-\mu)\ell$. By the minimum degree condition in G_r for every $v\in V$ there exists a $u\in U$ such that $(u,v)\in E(G_r)$. This in turn implies that among all triplets in T containing v at least $(\frac{3}{4}-\theta)^2$ proportion belongs to $N_A(u,T)$. Summing over all v we get:
$$|N_A(U,T)|>\left(\frac{3}{4}-\theta\right)^2|T|.$$
- Next, assume U is a subset of V with size $|U|=(\frac{3}{4}-\theta)^2(1-\mu)\ell$. By the minimum degree condition in G_r, any two vertices $v,w\in V$ have a common neighbor $u\in U$. This implies that among all the triplets in T containing v and w, at least a $\frac{3}{4}-\theta$ proportion belongs to $N_A(u,T)$. Once again, summing over all the pairs $v,w\in V$ we will have, that
$$|N_A(U,T)|>\left(\frac{3}{4}-\theta\right)|T|.$$

- Let $U \subset V$ with $|U| = (\frac{3}{4} - \theta)(1-\mu)\ell$. First we calculate the number of edges from U to the set of all possible pairs: $|U|(\frac{3}{4}-\theta)^2$. From this one can conclude that more than $\frac{5}{16}\binom{\ell}{2}$ pairs have more than $(\frac{1}{4}+\theta)\ell$ neighbors in U. None of those pairs can appear in any triplet of $\overline{N_A(U,T)}$. Besides, no pair can participate in more than $(\frac{1}{4}+\theta)\ell$ triplets in $\overline{N_A(U,T)}$. Thus, $|\overline{N_A(U,T)}| < \frac{1}{5}|T|$.
- Observing that any triplet in T has a neighborhood of size at least $(\frac{1}{4}-3\theta)\ell$ in V, we are done with proving the existence of a proportional matching.
- In order to finish the proof for the existence of a strong proportional matching, let us suppose that U has size $(1-\mu)\ell$. It is possible that triplets with degree 1 have no neighbor in U, but those are already matched to vertices in V. This implies that we have a strong proportional matching, too. ∎

For constructing our second bipartite graph $B = B(V,T,E(B))$ we are going to need the disjoint K_4-cover of G_r. Let $M \subset E(G_r)$ denote the edge set of this K_4-cover. $V = V(G_r)$ and $T = \{\{v_i,v_j,v_k\} : v_i,v_j,v_k \in V(G_r), (v_i,v_j) \in M(G_r)\}$, $E(B) = \{(u,\{v_i,v_j,v_k\}) : (u,v_i),(u,v_j),(u,v_k) \in E(G_r)\}$. We call v_k the third vertex of the triplet. Again, $0 < \theta \ll \mu \ll 1$.

Lemma 12. *There is a proportional matching in the bipartite graph B, and it allows a strong proportional matching with respect to μ.*

Proof. We check the "strong" König–Hall conditions.

- Any $v \in V$ has $(\frac{1}{2}-2\theta)\frac{3}{2}\ell$ edges of M as neighbors (i.e., connected to both endpoints of those edges). This implies that
$$|N_B(v,T)| \geq \left(\frac{3}{4}-\theta\right)\left(\frac{1}{2}-2\theta\right)|T|.$$
- Let $U \subset V$ with $|U| = (\frac{3}{4}-\theta)(\frac{1}{2}-2\theta)(1-\mu)\ell$. Now for any $v \in V$ there exists a $u \in U$ such that $(u,v) \in E(G_r)$. This vertex u will have at least a $\frac{1}{2}-2\theta$ proportion of the edges in M as neighbors, and all of them with v will form a triplet in T which is connected to u. Summing over all $v \in V$ we get
$$|N_B(U,T)| > \left(\frac{1}{2}-2\theta\right)|T|.$$
- Let $U \subset V$ with $|U| = (\frac{1}{2}-2\theta)(1-\mu)\ell$. We want to prove that $|N_B(U,T)| > (\frac{1}{2}+2\theta)|T|\frac{1}{1-\mu}$. Compute first the number of vertices having more than $(\frac{1}{4}+2\theta)\ell$ neighbors in U. A simple calculation shows that this number is

"almost" $\frac{\ell}{2}$. Actually, it is less than $\frac{\ell}{2}$ in an extremal case, otherwise it is even bigger. Also, we have strictly more than $\frac{\ell}{2}$ of them if some "not too small proportion" have less neighbors than, say, 99% of U. Denote these V-vertices having big degree to U by S. Observe, that an S-vertex with an arbitrary other vertex has a common neighbor in U. Our second observation is that out of any three vertices at least two have a common neighbor in U. It simply follows from the minimum degree condition of G_r. Now, if in a K_4 there is exactly one S-vertex, the three edges adjacent to it are U-neighbors, and because of our second observation there is another edge having a U-neighbor. If there are two S-vertices in a K_4, then five edges will have some U-neighbor, and three or four S-vertices in a K_4 means that all the edges of it have some U-neighbor. Using these remarks a simple calculation shows, that if the S-vertices are not concentrated in "almost one half" of the 4-cliques of the cover, then strictly more than $\frac{2}{3}$ portion of the edges will have at least one U-neighbor. If an edge has a U-neighbor, then it cannot appear in more than $(\frac{1}{4}+\theta)\ell$ triplets of $\overline{N_B(U,T)}$. Hence, if the S-vertices are not concentrated in "almost one half" of the 4-cliques of the cover, then $|N_B(U,T)|$ will be strictly bigger than $\frac{|T|}{2}$, and this is what we want to prove. So, assume, that the S-vertices are "concentrated well enough" according to the K_4-cover. But there is one more thing we know about S: its size cannot be strictly bigger than $\frac{\ell}{2}$. In this case we will be done, since we would have more than $\frac{2}{3}$ portion of the edges having a U-neighbor. Summarizing our knowledge about S we know that S is concentrated in "almost" one half of the cliques of the K_4-cover, and almost all vertices in S have full degree to U. But an edge with endpoints having more common neighbors in U than $(\frac{1}{4}+\theta)\ell$ will constitute a U-neighboring triplet with any third vertex. Hence, even if the S-vertices are well concentrated, $|N_B(U,T)|$ will be big enough.

- Let $U\subset V$ with $|U|=(\frac{1}{2}+2\theta)\ell$. Then any edge in M will have a neighbor in U, thus we will have $(\frac{3}{4}-\theta)|T|$ triplets in $N_B(U)$.
- Let $U\subset V$ with $|U|=(\frac{3}{4}-\theta)(1-\mu)\ell$. First we calculate the number of edges from U to M: $|U|(\frac{1}{2}-2\theta)$. From this one can conclude that more than $\frac{1}{5}|M|$ edges have more than $(\frac{1}{4}+\theta)\ell$ neighbors in U. None of those edges can appear in any triplet of $\overline{N_B(U,T)}$. Besides, no edge can participate in more than $(\frac{1}{4}+\theta)\ell$ triplets in $\overline{N_B(U,T)}$. Thus, $|\overline{N_B(U,T)}|<(\frac{1}{5}+\theta)|T|$.
- Observing that any triplet in T has a neighborhood of size at least $(\frac{1}{4}-3\theta)\ell$ in V, we are done with proving the existence of a proportional matching.

- To finish the proof for the existence of a strong proportional matching let us suppose that U has size $(1-\mu)\ell$. It is possible that triplets with degree 1 have no neighbor in U. But those are already matched to vertices in V. This implies the existence of a strong proportional matching. ∎

Next, we will present the procedure for assigning the vertices in I to G_r, starting with the vertices in $I_1 - I_1'$. Assume $\mathcal{M}_1$ denotes the proportional matching provided by Lemma 11 with respect to the graph A. For a cluster V_t, let $\{V_i, V_j, V_k\}$ be one of the triplets matched to it in $\mathcal{M}_1$. We will assign the vertices of $(I_1-I_1')\cap A(V_i)\cap A(V_j)\cap A(V_k)$ to the cluster V_t by adding them to the list $L(V_t)$. Using Lemma 10, $|(I_1-I_1')\cap A(V_i)\cap A(V_j)\cap A(V_k)|$ is almost the same for all choices of $\{V_i, V_j, V_k\}$, which in turn implies that the list $L(V_t)$ for all $V_t \in G_r$ will have almost the same length after the distribution of $I_1 - I_1'$. Also, note that the construction of $A(G_r)$ and the structure of the proportional matching $\mathcal{M}_1$ implies that if $x \in I_1 - I_1'$ is assigned to $L(V_t)$ then the $N_H(x)$-vertices are assigned to neighboring clusters of V_t. The distribution of $I_2 - I_2'$ also can be carried out in the same manner using the matchings provided by Lemmas 11 and 12.

For the vertices in I_3, since all their neighbors in H' are assigned to a single triangle in the K_4-cover of G_r, we can assign them to the fourth cluster of the same 4-clique. By Lemma 10 almost the same number of such vertices will be assigned to each cluster in G_r.

We remark that there are other cases to consider: e.g., some of the independent vertices can have all their neighbors assigned to two clusters. But it is easy to see that those matchings are easy to find once the harder cases are dealt with. Then mapping such I-vertices can be done in a similar way as we did for others.

In the next section we will consider the assignment of I_1' and I_2' vertices. For this we will use both proportional and strong proportional matchings. As we will see, for assigning a proportion of these vertices, we will need to use the strong proportional matchings, and the leftover will be distributed among the clusters of G_r by the proportional matchings.

5.3. Preparations for the embedding

The embedding of H will follow a similar procedure to that of the Blow-up Lemma [8]. Observe that in the Blow-up Lemma the reduced graph is fixed, does not depend on the parameters, and all the edges of the reduced graph are super-regular pairs. Obviously, these conditions are too strong in our case. Besides, as we will see, there will be restrictions for the embedding of certain H-vertices. Hence, we have to make changes in the embedding procedure of the Blow-up Lemma.

5.3.1. Bad vertices in G Every vertex in a cluster of a super-regular pair has big degree to the other cluster. Our edges in G_r are regular pairs, some vertices may have just a small number of neighbors in the other cluster (this number can be even zero). To avoid problems which can be caused by this, we are going to discard some vertices from the clusters and put them into V_0. If a considerable portion of the independent vertices are in I_3 (say, more than $\frac{1}{3}|I|$), then we make all the edges of the K_4-cover super-regular. Removing at most $3\varepsilon m$ vertices from each cluster (those vertices, which have small degrees inside the K_4s), we put at most $3\varepsilon n$ new vertices in V_0. Otherwise, if I_3 is not big enough, then either I_1 or I_2 will be large enough, say I_1. Let $\mathcal{M}_1$ be the proportional matching provided by
KULemma11 for the clusters in G_r. For a fixed cluster $V_t \in V(G_r)$ let $\mathcal{T}$ denote the set of triplets matched to V_t in $\mathcal{M}_1$. We call a vertex $v \in V_t$ *bad*, if v has degree less than $d-\varepsilon$ to at least half of the triplets in $\mathcal{T}$, i.e., to at least one of the clusters of those triplets.

Lemma 13. *No cluster in G_r can contain more than $6\varepsilon m$ bad vertices.*

Proof. For a cluster $V_t \in V(G_r)$ which is matched to the triplets of $\mathcal{T}$, let $\{v_1, \dots, v_s\}$ denote the set of bad vertices. If $s > 6\varepsilon m$ then there should be a triplet in $\mathcal{T}$ to which more than $3\varepsilon m$ vertices of V have bad degrees. Thus to one of the clusters of this triplet there are more than εm vertices with degree less than $d-\varepsilon$, which contradicts the ε-regularity condition. ∎

By removing the $6\varepsilon m$ bad vertices from the cluster V_t we can guarantee that all its vertices have big degrees to at least half of the matched triplets, and overall at most $6\varepsilon n$ bad vertices will be added to V_0.

5.3.2. Preprocessing – finishing the proof of the Main Lemma Observe that the number of vertices participating in the clusters of G_r is about $(1-\theta)n$, which means that θn vertices from H have to be mapped to vertices of the exceptional cluster V_0. In order to guarantee that the embedding of these θn vertices of H can be carried out, we perform a preprocessing step. This preprocessing is done as follows: first we assign the necessary number of I_1'-vertices (or I_2'-vertices, if I_1' is too small) using the strong proportional matching, second, we perform a *switching* procedure. Since for every $1 \le i \le \ell$ $|L(V_i)| \ge |V_i|$, we form L_0 by removing a subset of I' vertices which are assigned by the "ordinary" proportional matching, so that for every $1 \le i \le \ell$, $|L(V_i)| = |V_i|$. Also, we put the cut-out vertices into L_0. Let $\phi : L_0 \to V_0$

be any bijective mapping. We need to ensure that the assignment of L_0 to V_0 is consistent with $E(H)$; that is, for any $x \in L_0$, with $(x,y_1),(x,y_2)$, and $(x,y_3) \in E(H)$, if $v = \phi(x)$, $y_1 \in L(V_i)$, $y_2 \in L(V_j)$, and $y_3 \in L(V_k)$ then $\deg_G(v,V_i)$, $\deg_G(v,V_j)$, and $\deg_G(v,V_k)$ are all at least $\frac{m}{10}$. If this condition does not hold for a pair (x,v), a switching will be performed. In the switching operation we first pick a cluster V_s and then locate a vertex x' in $L(V_s)$ such that (V_i,V_s), (V_j,V_s), (V_k,V_s) are all edges in $E(G_r)$. Furthermore, if $(x',y_1'),(x',y_2')$, and $(x',y_3') \in E(H)$ with $y_1' \in L(V_{i'})$, $y_2' \in L(V_{j'})$, and $y_3' \in L(V_{k'})$ then $\deg_G(v,V_{i'})$, $\deg_G(v,V_{j'})$, and $\deg_G(v,V_{k'})$ are all at least $\frac{m}{10}$. We will see that such x' can be found among the preprocessed vertices.

Lemma 14. *For every $x \in L_0$ there exists an x' as required above.*

Proof. Let us consider the case when the switching is done through I_2'-vertices; the case for I_1'-vertices is similar. It is easy to see that for $v \in V_0$, it has degree $\frac{m}{10}$ to at least 72% of all clusters. Let V_s be as above, the number of clusters in its neighborhood for which v has large degree ($\geq \frac{m}{10}$) is at least $\frac{46}{100}\ell$, which in turn will span at least $\frac{14}{100}\frac{3\ell}{2}$ edges from the K_4 cover of G_r. We can conclude that at least 6% of the edge-vertex triplets will be spanned by these clusters. Any edge-vertex triplet τ will allocate ω vertices during the preprocessing. This means V_s receives at least $\frac{\omega}{\ell}\mu$ from every triplet which is connected to it. This is the point where we use the strong proportional matching. The triplets spanned by the clusters in the common neighborhood of V_s and v are at least 6% of the edge-vertex triplets. Letting $\Omega = \frac{3\ell^2\omega}{2}$, the number of assigned vertices during the preprocessing to V_s determined by this 6% of all the triplets is at least $\frac{6}{100}\frac{\mu}{\ell}\Omega$. Finally, if the number of vertices assigned by the strong proportional matching to the common neighborhood of V_i, V_j, and V_k is at least $|V_0|$, then we can find x'. More precisely, we need that

$$\left(\frac{1}{4} - 3\theta\right)\ell\frac{\mu}{\ell}\frac{6\Omega}{100} \geq |V_0|.$$

This is certainly true for $\Omega \geq \frac{100}{\mu}|V_0|$. ∎

It should be pointed out that we can perform this switching procedure in such a way that the neighbors of the switched x's are scattered almost evenly in a constant proportion of the triplets, and so in a constant proportion of the clusters. That is, there is a constant K such that no cluster will contain more than $K\frac{|V_0|}{\ell}$ neighbors.

Observe, that with the above remark we have found such a decomposition of the vertices of H that satisfies Lemma 6, the Main Lemma.

We will next present the criterion for H to be non-extremal: $|I'_1 \cup I'_2|$ should be at least as big as $\frac{100}{\mu}\theta n$. What we need are $\frac{100}{\mu}|V_0|$ vertices for the strong proportional matching; the rest of $|I'_1 \cup I'_2|$ is assigned by the "ordinary" proportional matchings.

5.3.3. The embedding order of the H-vertices Now we have consistently assigned the vertices of $H - L_0$ to clusters of G_r, and we have already decided the actual embedding of the L_0-vertices (Main Lemma). Note, that the embedding of L_0-vertices will restrict us when embedding their neighbors. The rest of the embedding procedure will follow the same line as the Blow-up Lemma. We will choose the parameters ε', ε'', ε''',δ''', δ'', and δ' so that

$$\varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon''' \ll \delta''' \ll \delta'' \ll \delta' \ll d.$$

For simplicity we will assume that all the densities of the regular pairs are d, there are no larger densities. This can be achieved by randomly discarding edges with the appropriate probabilities from the denser regular pairs. For every $1 \le i \le \ell$, the vertices of L_i will be embedded in the cluster V_i. Let $n' = |V(H - L_0)|$, we order the vertices of $H - L_0$ into a sequence $\mathcal{S} = (x_1, x_2, \ldots, x_{n'})$ which is almost the order in which $V(H - L_0)$ will be embedded. For each $1 \le i \le \ell$, choose a subset B_i of L_i of size $\delta' m$ from I' (these are at distance of at least 4 from each other). The elements of the B_is will be called buffer vertices. Observe that by Lemma 10 the vertices $B = \cup_i B_i$ have almost the same number of neighbors in every list L_j, $1 \le j \le \ell$. Let $M = |B|$, and $b_1, b_2, \ldots, b_M$ be the buffer vertices, then they will form the last part of $\mathcal{S}$. The sequence $\mathcal{S}$ starts with the vertices of $N_H(L_0)$, followed by $\{N_H(b_1),\ N_H(b_2), \ldots, N_H(b_M)\}$. We let $T_0 = \sum_{i=1}^{M} |N_H(b_i)|$. Then we add all the other vertices to the sequence, in such a way that the buffer vertices form the tail of $\mathcal{S}$. For technical reasons we assume that $\mathcal{S}$ is ordered evenly according to the L_i lists, i.e., the consecutive segments of length $\delta'' n'$ have the same number of vertices from every list. Later we may place some vertices forward, but then we rearrange $\mathcal{S}$ to maintain this property.

5.4. The embedding algorithm

The embedding of the vertices of $H - L_0$ occurs in three separate phases. In the first two phases we are going to embed the vertices of $N_H(L_0)$, and then will come the embedding of the next vertices of $\mathcal{S}$ after each other according to their position in the sequence (some reordering is possible), until only buffer vertices are left in $\mathcal{S}$. In the third phase, by a matching procedure we embed the remaining buffer vertices. Let us mention that the phase for embedding $N_H(L_0)$ is a randomized procedure, while the other two are deterministic.

In the next subsection we outline our method for the embedding, with the exception of selecting a vertex to be covered. That will be done in a separate subsection.

5.4.1. Outline For an unembedded vertex $x \in L_i$ let $H_{t,x}$ denote its monotonically shrinking *host set* in V_i at time t. Also, for technical reasons we keep track of another set, $C_{t,x}$. By Z_t we denote the set of occupied vertices (note that $Z_0 = V_0$), and we also maintain a set Bad_t of exceptional pairs in $H - L_0$.

At time 0, we set $C_{0,x} = H_{0,x} = V_i$, where $x \in L_i$, and x does not have any neighbor in L_0. For those vertices having neighbors in L_0 the setup is different. Let x in L_0 have neighbors $y_1 \in L_i$, $y_2 \in L_j$ and $y_3 \in L_k$, and $v = \phi(x)$. By the end of the switching process, we have ensured that v has at least $\frac{m}{10}$ neighbors in V_i, V_j and V_k. These neighborhoods give $C_{0,y_1} = H_{0,y_1}$, $C_{0,y_2} = H_{0,y_2}$ and $C_{0,y_3} = H_{0,y_3}$, respectively.

Recall, that $T_0 = \sum_{i=1}^{M} |N_H(b_i)|$. We let $T_2 = \delta'' n'$ and $T_1 = |N_H(L_0)|$. Given the initial host sets, the embedding algorithm will go as follows:

Phase 0. For $1 \le t \le T_1$ repeat the following steps

Pick an appropriate vertex v_t for $x_t \in N_H(L_0)$ randomly and uniformly from H_{t-1,x_t} using the *Selection Algorithm*.
Update

$$Z_t = Z_{t-1} \cup \{v_t\}$$

and for all unembedded vertices x_i, with $t < i \le n'$

$$C_{t,x_i} = \begin{cases} C_{t-1,x_i} \cap N_G(v_t) & \text{if } (x_i, x_t) \in E(H), \\ C_{t-1,x_i} & \text{otherwise,} \end{cases}$$

and

$$H_{t,x_i} = C_{t,x_i} - Z_t$$

Phase 1. For $t \ge T_1 + 1$ repeat the following steps

Step 1. Embed the vertex x_t from the sequence $\mathcal{S}$. Using the *Selection Algorithm* choose an appropriate vertex v_t from the set H_{t-1,x_t} as x_t's image.
Step 2. Update

$$Z_t = Z_{t-1} \cup \{v_t\}$$

and for all unembedded vertices x_i, with $t<i\leq n'$

$$C_{t,x_i}=\begin{cases} C_{t-1,x_i}\cap N_G(v_t) & \text{if } (x_i,x_t)\in E(H),\\ C_{t-1,x_i} & \text{otherwise,}\end{cases}$$

and

$$H_{t,x_i}=C_{t,x_i}-Z_t$$

Step 3. Exceptional vertices in G

1. If $t\neq T_0+T_1$ go to step 4.

2. If $t=T_0+T_1$ then for every cluster V_i form a set E_i containing those uncovered vertices satisfying

$$|\{b: b\in B_i, v\in C_{t,b}\}|<\delta''|B_i|.$$

We will cover them right after the neighbors of the buffer vertices, thereby eliminating a possible objection to embed the buffer vertices in *Phase 2.* We slightly change the ordering of $\mathcal{S}$. From every list $L(V_i)$ we take $|E_i|$ vertices belonging to I' and place them forward. We will maintain the even ordering of $\mathcal{S}$, which is possible since for all i, E_i is a small set, as we will see later. The choice of these vertices is not arbitrary. First, these are from I', because this way they do not disturb the neighborhood of each other and the neighborhoods of the buffers. Also, we cover the exceptional v vertex by such an I'-vertex, which has all its neighbors assigned to clusters to which v has big enough degree. Moreover, we require that the neighbors of the vertices we use to cover in this step are "well spread" among the clusters of G_r, i.e., no cluster will have more such neighbors than $K\varepsilon''m$ (here K is a constant). We will prove later that this is possible to achieve.

Step 4. Exceptional vertices in $H-L_0$

1. If T_2 does not divide t, then go to Step 5.

2. If T_2 divides t, we will find all exceptional unembedded vertices $y\in H-L_0$ such that $|H_{t,y}|\leq(\delta')^2m$. We again slightly change the order of the remaining vertices in $\mathcal{S}$ by bringing these exceptional vertices forward in $\mathcal{S}$ (including the exceptional buffer vertices) and will maintain the even distribution of vertices assigned to different clusters. This is possible because of the very small number of exceptional vertices we can find in this step.

Step 5. If the unembedded vertices are all buffer vertices, go to *Phase 2.*, otherwise set $t\leftarrow t+1$ and go back to *Step 1.*

Phase 2. Find a system of distinct representatives of the sets $H_{t,y}$ for all unembedded vertices.

5.4.2. Selection Algorithm There can be two possible cases.

Case 1. $x_t \notin E_H$.
As the image of x_t, we will choose some $v_t \in H_{t-1,x_t}$ such that the following conditions are satisfied for every unembedded vertex y with $(x_t, y) \in E(H)$:

$$(d-\varepsilon)|H_{t-1,y}| \le \deg_G(v_t, H_{t-1,y}) \le (d+\varepsilon)|H_{t-1,y}|, \tag{1}$$

$$(d-\varepsilon)|C_{t-1,y}| \le \deg_G(v_t, C_{t-1,y}) \le (d+\varepsilon)|C_{t-1,y}|, \tag{2}$$

and

$$(d-\varepsilon)|C_{t-1,y} \cap C_{t-1,y'}| \le \deg_G(v_t, C_{t-1,y} \cap C_{t-1,y'}) \le (d+\varepsilon)|C_{t-1,y} \cap C_{t-1,y'}|, \tag{3}$$

for at least $(1-\varepsilon')$ portion of the unembedded vertices y' so that y and y' are assigned to the same cluster V_i, and $\{y, y'\} \notin Bad_{t-1}$. The set Bad_t will be formed as the union of Bad_{t-1} and those pairs $\{y, y'\}$ which does not satisfy (3) for v_t. Clearly, at most $3\varepsilon' m$ new vertices will be added to Bad_t.

Case 2. $x_t \in E_H$.
As explained in the second part of *Step 3*, we can assign $x_t \in L(V_i)$ to an exceptional $v_t \in E_i$ so that for all unembedded $y \in N_H(x_t)$ the following is satisfied:

$$\deg_G(v_t, C_{t-1,y}) = \deg_G(v_t) \ge (d-\varepsilon)m \ge (d-\varepsilon)|C_{t-1,y}|, \tag{4}$$

and

$$\deg_G(v_t, H_{t-1,y}) \ge \deg_G(v_t) - 3\delta' m - |E_i| \ge (d-\varepsilon)m - 4\delta' m \ge \frac{d}{2}m. \tag{5}$$

In (5) we use the fact that for each i $|E_i| \le \delta' m$. We will prove this in Lemma 18.

5.5. Correctness

We start by proving that *Phase 0* of the algorithm succeeds with high probability. First we show that the Selection Algorithm succeeds for $1 \le t \le T_1$ in finding the v_t-vertices.

Lemma 15. *Assuming that Phase 0 succeeds for all t', with $t' < t \le T_1$ and $H_{t-1,x_t} \ge \delta'' m$, then it succeeds for t.*

Proof. We only need to consider **Case 1** of the *Selection Algorithm*. The selected vertex $v_t \in H_{t-1,x_t}$ should satisfy conditions (1), (2), and (3). By the ε-regularity we will have at most $2\varepsilon m$ vertices in H_{t-1,x_t} which do not satisfy (1), and the same holds for (2). For condition (3) we will define a bipartite graph $B=(W_1,W_2,E(B))$. Here $W_1=H_{t-1,x_t}$, and the elements of W_2 are the sets $C_{t-1,y}\cap C_{t-1,y'}$ for all pairs $\{y,y'\}$ where $(x_t,y)\in E(H)$, y and y' are both assigned to the same cluster, and $\{y,y'\}\notin Bad_{t-1}$. For $v\in W_1$ and $u\in W_2$, we have $(v,u)\in E(B)$ if (3) does not hold for v and the pairs corresponding to u. If we assume that there are more than $\varepsilon' m$ vertices $v\in W_1$ with $\deg_B(v)>\varepsilon'|W_2|$, then there should be a vertex $u\in W_2$ such that

$$\deg_B(u) > \varepsilon'^2 m \gg \varepsilon m.$$

But this is a contradiction with the ε-regularity since the pair $\{y_u,y'_u\}$ corresponding to u does not belong to Bad_{t-1} and

$$|C_{t-1,y_u}\cap C_{t-1,y'_u}| \geq (\delta-\varepsilon)^6 m \gg \varepsilon m.$$

This in turn implies that H_{t-1,x_t} can contain at most $4\varepsilon m+\varepsilon' m \ll \delta'' m$ vertices which cannot be used to map x_t, proving the succession of *Phase 0.* ∎

Observe, that when we progress to *Phase 1* (after the successful completion of *Phase 0*), the aformentioned proof will work. We will be able to find a vertex to cover if the host sets are not too small.

What is left to show is that for all time t, $1\leq t\leq T_1$, the host sets do not become too small. Actually, we prove this not just for the host sets for the unembedded $N_H(L_0)$-vertices, but for all unembedded H-vertices.

Lemma 16. *If Phase 0 succeeds for all t, with $t<T_1$ then for all $t'>t$ $H_{t-1,x_{t'}}\geq\delta'' m$ with high probability.*

Proof. For $x\in N_H(L_0)$ $|H_{0,x}|\geq\frac{m}{10}$. In Lemma 15 we proved that if the algorithm succeeds up to time t and $|H_{t,x}|\geq\delta'' m$, we can find a v_t to embed x. Since no two vertices in $N_H(L_0)$ are adjacent, the only way the host set of x decreases is that we cover some vertices of it by other $N_H(L_0)$-vertices. When deciding which G-vertex to cover by an $N_H(L_0)$-vertex x, the Selection Algorithm will always provide almost all of $H_{t,x}$ as a possibility – a subset of size $\delta'' m$ can be left out. Out of those possibilities we choose the host vertex randomly and uniformly. Recall that according to our remark, the $3|V_0|$ restricted vertices of $N_H(L_0)$ are distributed among a constant proportion of the clusters, as evenly as possible. We can conclude that only a very small number of vertices ($constant\cdot\theta m$) have a restriction to embed them into a set of size at least $\frac{m}{10}$ in each cluster. Now, by applying the statement of

Lemma 15 one can easily conclude that up to time T_1 all the unembedded $N_H(L_0)$-vertices have a host set of size at least $\frac{m}{11}$.

A vertex from the rest of H can have three neighbors embedded by time T_1. This means these sets may shrink up to three times, each time the sizes are multiplied by a number between $d-\varepsilon$ and $d+\varepsilon$. We also lose some places because of the embedded $N_H(L_0)$-vertices. The randomness in the *Selection Algorithm* helps us. The expected number of covered vertices in a host set is proportional to its size. The choices for picking a host vertex for an $N_H(L_0)$-vertex are “almost independent”. Let x be unembedded at time T_1, and denote the expected number of covered vertices in the set $C_{T_1,x}$ by $E(T_1,x)$. Denote the actual number of covered vertices by $A(T_1,x)$. It is an easy exercise to show that

$$Pr(A(T_1,x) - E(T_1,x) > O(m^{\frac{3}{4}})) < exp(-O(\sqrt{m})).$$

Noting that we have linear number of host sets in a cluster, we get that with high probability, for every unembedded vertex x at time t, $1 \le t \le T_1$, $|H_{t,x}| \ge d^4 m$. ∎

One can easily conclude from the above that *Phase 0* succeeds with probability $1-o(1)$.

For $t > T_1$ we will need a more thorough analysis. At time t for the cluster V_i and a subset of the unembedded vertices $L_i \subseteq L(V_i)$, we define a bipartite graph $U_t = (V_i, L_i, E(U_t))$. Here if $x \in L_i$, $v \in V_i$, and $v \in C_{t,x}$ then $(x,v) \in E(U_t)$.

The following lemma is pivotal for the proof of the correctness of *Phase 1*.

Lemma 17. *For every $1 \le i \le \ell$ and $T_1 + 1 \le t \le T_0 + T_1$ and any set of unembedded vertices $L_i \subseteq L(V_i)$ at time t, with $|L_i| \ge (\delta''')^2 m$, if Phase 1 succeeds for all $t' \le t$, then apart from an exceptional set F of size at most $\varepsilon'' m$ the following will hold for every $v \in V_i$:*

$$\deg_{U_t}(v) \ge (1-\varepsilon'')d(V_i, L_i)|L_i|.$$

Proof. We use the so called “defect form” of the Cauchy–Schwarz inequality, that states: if for some $p \le q$

$$\sum_{i=1}^{p} \alpha_i = \frac{p}{q}\sum_{i=1}^{q} \alpha_i + \beta$$

then

$$\sum_{i=1}^{q} \alpha_i^2 \ge \frac{1}{q}\left(\sum_{i=1}^{q} \alpha_i\right)^2 + \frac{\beta^2 q}{p(q-p)}.$$

Assume to the contrary that the lemma is not true that is $|F| > \varepsilon'' m$. Choose $F_0 \subset F$ with $|F_0| = \varepsilon'' m$. Define $\nu(t,x)$ as the number of embedded neighbors of x by time t. Observe that if x has a neighbor in L_0 or in $N_H(L_0)$, then $\nu(0,x) \geq 1$, otherwise it is 0. Then

$$|E(U_t)| = \sum_{x \in L_i} |C_{t,x}| \geq \sum_{x \in L_i} (d-\varepsilon)^{\nu(t,x)} m. \tag{6}$$

We also have

$$\begin{aligned} &\sum_{x \in L_i} \sum_{x' \in L_i} |C_{t,x} \cap C_{t,x'}| \\ &\leq \sum_{x \in L_i} \sum_{x' \in L_i} (d+\varepsilon)^{\nu(t,x)+\nu(t,x')} m + |L_i| m + 12|L_i| m + 3\varepsilon' m^3 \\ &\leq \sum_{x \in L_i} \sum_{x' \in L_i} (d+\varepsilon)^{\nu(t,x)+\nu(t,x')} m + 4\varepsilon' m^3 \end{aligned} \tag{7}$$

For each pair $\{x,x'\}$, we can upper-bound $|C_{t,x} \cap C_{t,x'}|$ by m. The diagonal terms $(x = x')$ result in error $|L_i| m$. For the non-diagonal terms for which $N_H(x) \cap N_H(x') \neq \emptyset$ we have the term $6|L_i|m$. If $\{x,x'\} \in Bad_t$, by **Case 1** of the Selection Algorithm either x or x' can appear in at most $3\varepsilon' m$ bad pairs. Hence there will be at most $3\varepsilon' m^2$ bad pairs associated with the cluster V_i. Using the Cauchy–Schwarz inequality with $p = \varepsilon'' m$, $q = m$ and the variables $\alpha_k = \deg_{U_t}(v_k)$, $1 \leq k \leq m$ with $v_k \in V_i$ and the first $\varepsilon'' m$ values set to degrees in F_0, we have:

$$\begin{aligned} |\beta| &= \varepsilon'' \sum_{v \in V_i} \deg_{U_t}(v) - \sum_{v \in F_0} \deg_{U_t}(v) \\ &\geq \varepsilon'' \sum_{v \in V_i} \deg_{U_t}(v) - \varepsilon''(1-\varepsilon'') d(V_i, L_i) |L_i| m \\ &= (\varepsilon'')^2 \sum_{v \in V_i} \deg_{U_t}(v). \end{aligned} \tag{8}$$

Then using (6), (8) and the Cauchy–Schwarz inequality we get

$$\begin{aligned} \sum_{x \in L_i} \sum_{x' \in L_i} |C_{t,x} \cap C_{t,x'}| &= \sum_{v \in V_i} \left(\deg_{U_t}(v)\right)^2 \\ &\geq \frac{1}{m} \left(\sum_{v \in V_i} \deg_{U_t}(v) \right)^2 + (\varepsilon'')^3 d(V_i, L_i)^2 m |L_i|^2 \\ &\geq \frac{1}{m} \left(\sum_{x \in L_i} (d-\varepsilon)^{\nu(t,x)} m \right)^2 + (\varepsilon'')^3 (d-\varepsilon)^6 m |L_i|^2 \\ &\geq \sum_{x \in L_i} \sum_{x' \in L_i} (d-\varepsilon)^{\nu(t,x)+\nu(t,x')} m + (\varepsilon'')^3 (d-\varepsilon)^6 m |L_i|^2 \end{aligned}$$

which is a contradiction to (7), since $|L_i| \geq (\delta''')^2 m$,

$$(\varepsilon'')^3 (d-\varepsilon)^8 (\delta''')^2 \gg 4\varepsilon' \gg 4\varepsilon$$

and

$$(d+\varepsilon)^{\nu(t,x)+\nu(t,x')} - (d-\varepsilon)^{\nu(t,x)+\nu(t,x')} \ll 4\varepsilon. \qquad \blacksquare$$

As a consequence we will have the following bound on the size of the exceptional sets E_i:

Lemma 18. *In Step 3, for each $1 \leq i \leq \ell$ we have $|E_i| \leq \varepsilon'' m$.*

Proof. Applying the previous lemma with $t = T_0$ and $L_i = B_i$, which means $|L_i| \geq (\delta''')^2 m$, we will have

$$(1-\varepsilon'') d(V_i, L_i)|L_i| \geq \frac{d^3}{2}|L_i| > \delta''|L_i|$$

and $E_i \subset F$. $\blacksquare$

In the next lemma we will prove the succession of the embedding algorithm for covering the exceptional G-vertices.

Lemma 19. *In Step 3 for each $1 \leq i \leq \ell$ and $v \in E_i$ we can find an unembedded $x \in L(V_i)$ to cover v. This $x \in I'$, and its neighbors are assigned to such clusters to which v has degree at least $\frac{d}{2}m$. Also, the (assigned, but not embedded) neighbors of these xs are well spread among the clusters of G_r, no cluster will have more than $6\varepsilon'' m$.*

Proof. If we have a considerable portion of I_3-vertices, then the lemma trivially follows. There are two cases left: the I_1 and the I_2-vertices. We will discuss the case of I_2-vertices, i.e., when most of the independent vertices are from I_2. (The case of I_1-vertices is very similar.)

Denote the proportional matching of this case by M. For a cluster V_i let $\mathcal{T}_i$ denote the set of triplets matched to it in M. Recall that we removed the *bad* vertices from every cluster (Lemma 13). Hence, at time 0 all the vertices had degree more than $d-\varepsilon$ to at least half of the triplets in $\mathcal{T}_i$, i.e., to all clusters of those triplets.

We are at time $T_0 + T_1$ now, after embedding $N_H(L_0)$ and the neighborhood of the buffer vertices. Note that we payed attention to embed these H-vertices as evenly as possible. Not just the neighbors of the buffers are well spread, but $N_H(L_0)$ as well. Hence, even at time $T_0 + T_1$ every vertex in the cluster V_i has degrees big enough to at least 50% of the triplets of $\mathcal{T}_i$ (here we used the fact that the number of buffer vertices is very small, and $N_H(L_0)$ is embedded randomly). The exceptional sets of the clusters are small, as we showed in Lemma 18, hence, there are enough candidates to choose.

Now we prove that the covering of the G-vertices can be done in such a way, that the neighbors of the embedded vertices will not concentrate in any of the clusters.

First we will show a simple and easy-to-analyze algorithm for the ideal case when every vertex in V_i to be covered is connected to all the triplets of $\mathcal{T}_i$ for all i. Then we will modify it for our more general setup.

The algorithm is as follows: We start by completing all the E_i sets by arbitrary V_i-vertices obtaining equal size sets. Consider the triplets of $\mathcal{T}_1$, $\tau_1, \dots, \tau_k$ $(k = \frac{3\ell}{2})$. Pick one-one unembedded vertex from I_2' which is assigned by these triplets, and cover k E_1-vertices with those I_2'-vertices. Then repeat it for all $\mathcal{T}_i$. When we are ready, we have finished one *round*. It is easy to see that in one round we covered exactly k vertices from each exceptional set, and because all cluster appears in the same number of triplets, in 4.5ℓ, after these coverings we have the same number of neighbors in every cluster, this number is then 4.5ℓ. Iterating the rounds at the end we will arrive to the situation when no E_i-vertices are left, and for every cluster the embedded vertices have the same number of assigned neighbors, which is $3\varepsilon'' m$.

Let us return to the assumptions of the lemma. The modified algorithm for the general case will be different in two points. We again start by completing the sets to the same size by adding arbitrary vertices from the corresponding cluster . We take the triplets of the matching one by one, as we did previously. But we cannot always find a vertex to be covered for a triplet. In such a case, we take the next triplet. Even so in every round at least half of the triplets will assign a vertex which will cover a G-vertex, and the embedded vertices will have at most 4.5ℓ assigned neighbors in a cluster. Iterate this procedure, and stop, when a set gets empty. If no exceptional vertices are left (vertices from the original E_i sets before adding other vertices), the algorithm stops. If not, then complete the sets to have the same size, and restart. Note, that now every exceptional set is at most half the size of the beginning. Hence, with $O(\log m)$ restarts no exceptional vertices are left uncovered. Since between two restarts the number of exceptional vertices is cut in half, and the "fastest decreasing" set has speed at most twice that of the slowest, we get no cluster that will have more than $6\varepsilon'' m$ embedded neighbors.
It is an easy exercise to check, that the case of I_1'-vertices is very similar. ∎

Next we will prove a result similar to Lemma 17 for $t > T_0 + T_1$.

Lemma 20. *For every $1 \le i \le \ell$ and $T_0 + T_1 < t \le T$ and any set of unembedded vertices $L_i \subseteq L(V_i)$ at time t, with $|L_i| \ge (\delta''')^2 m$, if Phase 1 succeeds* *for all $t' \le t$, then apart from an exceptional set of size at most $\varepsilon''' m$ the following will hold for every $v \in V_i$:*

$$\deg_{U_t}(v) \ge (1-\varepsilon''')d(V_i, L_i)|L_i|.$$

Proof. The proof follows the same line of argument as Lemma 17 with parameter ε''', except those vertices in the neighborhood of E_H. The inequality in (6) will hold with the same parameters, since for all $x \in N_H(E_H)$ we have

$$|C_{t,x}| \ge (d-\varepsilon)^{\nu(t,x)} m.$$

In (7) there are more bad pairs. More precisely, based on *Step 3* of the embedding algorithm, there will be an additional error term of $6\varepsilon'' m^2 |L_i|$. Using the fact that

$$(\varepsilon''')^3 (d-\varepsilon)^8 (\delta''')^2 \gg \varepsilon''$$

we can see that the contradiction still holds. ∎

The following lemma is an easy consequence of Lemmas 17 and 20.

Lemma 21. *For every $1 \le i \le \ell$ and $T_1 < t \le T$ and any set of unembedded vertices $L_i \subseteq L(V_i)$ at time t, with $|L_i| \ge (\delta''')m$ and a set $A \subset V_i$ with $|A| \ge (\delta''')m$, if Phase 1 succeeds for all $t' \le t$ then apart from an exceptional set F of size at most $(\delta''')^2 m$, the following will hold for every $x \in L_i$:*

$$|A \cap C_{t,x}| \ge \frac{|A|}{2m}|C_{t,x}|.$$

Proof. Let us suppose that the lemma is not true, there exists a set $F \subseteq L_i$ such that $|F| > (\delta''')^2 m$, and for every $x \in F$ the inequality of the statement does not hold. We again consider the bipartite graph $U_t = U_t(F, V_i)$.

$$\sum_{v \in A} \deg_{U_t}(v) = \sum_{x \in F} |A \cap C_{t,x}| < \frac{|A|}{2m} d(F, V_i)|F| m.$$

Applying Lemmas 17 or 20 with F, we get

$$\sum_{v \in A} \deg_{U_t}(v) \ge (1-\varepsilon''')d(F, V_i)|F|(|A| - \varepsilon''' m),$$

which is a contradiction. ∎

In the following lemma we show that the host sets do not become too small.

Lemma 22. *For every $T_1+1 \leq t \leq T$ and for every H-vertex y which is unembedded at time t, if Phase 1 succeeds for all $t' \leq t$ then the following holds:*

$$|H_{t,y}| > \delta'' m.$$

Proof. Let L_i be the set of all the unembedded vertices in V_i at time t, and let $A_t = V_i - Z_t$. Applying Lemma 21 we can see that for all $x \in L_i$ (except at most $(\delta''')^2 m$ vertices)

$$|H_{t,x}| = |A_t \cap C_{t,x}| \geq \frac{|A_t|}{2m}|C_{t,x}| \geq \frac{\delta'}{4}(d-\varepsilon)^3 m \gg (\delta')^2 m,$$

if $|A_t| \geq \frac{\delta'}{2}m$. Next we prove this statement. Let us suppose indirectly that there is a T' such that $T_1+1 \leq T' < T$ and

$$|A_{T'}| \geq \frac{\delta'}{2}m \text{ but } |A_{T'+1}| < \frac{\delta'}{2}m.$$

We know that at any time t, where T_2 divides t, there are at most $(\delta''')^2 m$ exceptional unembedded vertices. Thus, up to time T' we can find at most

$$\frac{1}{\delta''}(\delta''')^2 m \gg \delta'' m$$

exceptional vertices. This implies that at time T' there are many more than $(\delta' - \delta'')m$ unembedded buffer vertices, thus, on the contrary, $|A_{T'+1}| \gg (\delta' - \delta'')m$. Note, that we also proved that $T \leq \ell m - \ell\delta' m + \ell\delta'' m$. Let us consider now an arbitrary $y \in L(V_i)$ unembedded at time t $(1 \leq t \leq T)$, and let $k\delta'' n' = kT_2 \leq t < (k+1)T_2$ for some $0 \leq k \leq T/T_2$. There are two cases to discuss:

Case 1. If y was not among the at most $(\delta''')^2 m$ exceptional vertices of *Step 4*, then

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^3 (\delta')^2 m - K,$$

where K is the number of vertices covered in V_i during the period between kT_2 and $(k+1)T_2$. Recall that the sequence $\mathcal{S}$ is as balanced as possible; hence, $K \leq 2\delta'' m$, where $2\delta'' m$ comes from the reordering of $\mathcal{S}$ because of the exceptional vertices of G and H. Also, at time kT_2 we had that $|H_{kT_2,y}| \geq (\delta')^2 m$. These facts imply that in this case the statement of the lemma holds.

Case 2. If y was among the at most $(\delta''')^2m$ exceptional vertices of *Step 4*, then

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^3 (\delta')^2 m - K',$$

where K' is the number of vertices covered in V_i during the period between $(k-1)T_2$ and $(k+1)T_2$. Now K' can be as big as $(\delta'' + (\delta''')^2)m$, because at time $(k-1)T_2$ at most $(\delta''')^2m$ exceptional vertices were placed forward. Again, by observing that at time $(k-1)T_2$ we had that $|H_{(k-1)T_2,y}| \geq (\delta')^2m$, the proof of the lemma is finished. ∎

Now it is easy to show the succession of the *Selection Algorithm* in finding the v_t-vertices. We have just proved that the host sets can never get too small. In Lemma 15 we proved that *Phase 0* succeeds for time t, whenever it succeeds for all t' with $t' < t \leq T_1$ and the host set is big enough. It is easy to check that exactly the same proof works for *Phase 1* and up to time T. Putting these together, we have that *Phase 1* of the algorithm succeeds.

To prove that *Phase 2* of the algorithm succeeds, we will show that for all $1 \leq i \leq \ell$ there is a system of distinct representatives between the unembedded vertices of $L(V_i)$ and the remaining buffer vertices of V_i. Let $L_i \subset L(V_i)$ denote the set of unembedded vertices assigned to the cluster V_i, and $Y_i \subset V_i$ be the remaining vertices of the cluster V_i, with $M_i = |L_i| = |Y_i|$. Then by Lemma 22 for every $x \in L_i$ we will have $H_{T,x} > \delta''' M_i$. Furthermore, for all subsets $S \subset L_i$, if $|S| \geq \delta''' M_i$ then by Lemma 20

$$\left| \bigcup_{x \in S} H_{T,x} \right| \geq (1 - \delta''')M_i.$$

Finally, for any $v \in Y_i$, since v cannot be exceptional in G, by *Step 3* there are at least $\delta''' M_i$ host sets $H_{T,x}$ containing v implies that for the subsets $S \subset L_i$ with $|S| \geq (1 - \delta''' M_i)$ we have

$$\left| \bigcup_{x \in S} H_{T,x} \right| = M_i,$$

which in turn implies the existence of the system of distinct representatives. This finishes the proof in the non-extremal case. ∎

6. H is extremal and G is non-extremal

In this case we are assuming that apart from at most $\frac{100}{\mu}\theta n$ vertices of $\widetilde{H} \subset H$, the rest of $V(H)$ belong to disjoint K_4s.

Let us give the outline of the embedding in this case. First, we embed $\widetilde{H}$ in a similar way to what we did previously for a non-extremal H. We find the structure provided by the Second Decomposition Theorem, and stop before distributing V_0 (because we need the ε-regular edges). Using Lemmas 11 and 12 we almost evenly distribute $\widetilde{H}$ among the clusters of G_r. Then we embed $\widetilde{H}$ by our procedure similar to the Blow-up Lemma. Notice that now we have a lot of room because $\widetilde{H}$ is very small, thus we do not have to be careful, e.g., we do not need buffer vertices. Then we distribute V_0 among the 4 and 5-cliques of the cover (if the degrees are big enough). Note that the exceptional V_0-vertices may lose some neighbors, so we have to be careful when embedding $\widetilde{H}$. If $|\widetilde{H}| > \frac{dn}{2}$, then we randomly choose $\frac{2}{\ell}|\widetilde{H}|$ vertices from every cluster of G_r, and we embed $\widetilde{H}$ into those sub-clusters. The leftover vertices of these sub-clusters we put back to the original clusters. If ε is small enough then we will have the necessary "regularity" among these sub-clusters for this embedding, although, instead of ε we will have a parameter at most $\frac{\varepsilon}{d}$. Now, because we choose a random subset for the embedding, we can assume that every exceptional V_0-vertex has degree big enough – we lose the neighbors proportionally to the size of $\widetilde{H}$. If $|\widetilde{H}| \le \frac{dn}{2}$ then we do not have any problem: just notice that $d \ll \rho$, thus, we don't even need random subsets. Let us assume now, that we are after the embedding of $\widetilde{H}$ and we distributed the V_0-vertices appropriately to achieve the structure described in the Second Decomposition Theorem (we refer to the proof sketch of this theorem where the distribution of V_0-vertices was explained). We claim that this subgraph G' can be covered by disjoint 4-cliques. It is an easy consequence of the Blow-up Lemma that we can cover the 4-blocks by 4-cliques: all the edges within a 4-block are super-regular pairs, and every 4-block is perfectly balanced. For finishing the K_4-cover we use the argument of [1] (there it was used to give an alternative proof for the Hajnal–Szemerédi theorem for a non-extremal G). We repeat this short and elegant proof here.

Lemma 23. *Let $K_{h_1,\dots,h_5}$ be a complete 5-partite graph with color classes of sizes $h_1,\dots,h_5$, respectively. If $h_1+\dots+h_5=4m$ and $h_i \le m$ for all i, then $K_{h_1,\dots,h_5}$ can be covered by m vertex disjoint 4-cliques.*

Proof. We proceed by induction on m. Let us denote the color classes of $K_{h_1,\dots,h_5}$ by $V_1,\dots,V_5$, respectively. For $m=0$ there is nothing to prove, so let $m>0$. We may assume that $h_1 \le \ldots \le h_5$. First observe that the case $h_1=m$ is impossible, it implies that $h_1=\ldots=h_5$, hence $h_1+\ldots+h_5=5m$, i.e., $m=0$. Pick the first clique by choosing one-one vertex from the classes $V_2,\ldots,V_5$. Now we can apply the induction hypothesis, because $h_i \le m-1$ for all i. ∎

Applying Lemma 23 and the Blow-up Lemma we can find a 4-clique cover of a 5-block, if the sum of the cluster sizes is divisible by 4. For the "bad" 5-blocks we are going to use the connectivity structure among the 5-blocks, provided by the Second Decomposition Theorem. Recall, that the auxiliary graph F is connected. Let $\mathcal{B}_i$ be a bad block, then there is at least one more bad block $\mathcal{B}_j$ in the cover. This follows from the fact the the number of the vertices is divisible by 4. Denote the closest bad 5-block in F by $\mathcal{B}_j$. Consider any $(\mathcal{B}_i,\mathcal{B}_k)$ edge in F. It is easy to find a 4-clique using vertices from both blocks: if W is a cluster of a block having three regular edges to the clusters of the other block, then just take one-one vertex from W and the other three, which constitute a 4-clique. By extracting at most three 4-cliques with this method we can achieve that the sum of the cluster sizes of the block $\mathcal{B}_i$ is divisible by 4. On the other hand, we may destroy the divisibility of the other block. But repeating this clique extracting step starting from $\mathcal{B}_i$ going on the shortest path to $\mathcal{B}_j$, we can make all the blocks good in that path except possibly $\mathcal{B}_j$, the last block. So, we could decrease the number of bad blocks. Note, that there cannot be exactly one bad 5-block, this contradicts with the divisibility by 4. Hence, in a finite number of steps, after extracting a finite number of K_4s we will have only good 5-blocks. This finishes the proof of the embedding into a non-extremal G. ∎

7. *H* and *G* are extremal

We have arrived at the point when we need all the edges incident to the G-vertices. So far we could have managed to find the embedding even with all the degrees equal to $\frac{3}{4}n-o(n)$, but now we need the full strength of the minimum degree condition of G. We start by preparing G: if $\deg(v)$ and $\deg(w)$ are both greater than $\frac{3n-1}{4}$, and $(v,w)\in E(G)$, then delete that edge.

The rough outline of the embedding in this case is as follows. There is a quasi-independent set $P\subset V(G)$ of size $\frac{n}{4}$. Either we can find the embedding of H into G in such a way that almost all the vertices of $I(H)$ are in P, and the rest is embedded into $V(G)$–P, or we can find another quasi-independent set $Q\subset (V(G)–P)$ of size $\frac{n}{4}$. Then either we can embed almost all the vertices of the maximum independent set in $H-I(H)$ into Q, I into P, and the rest to $V(G)-(P\cup Q)$, or we can find a quasi-independent set $R\subset V(G)-(P\cup Q)$ of size $\frac{n}{4}$. In the latter case, we can find the embedding easily. In the following we sometimes write n in the form $4t+r$, where $0\le r\le 3$.

7.1. Case 1: one quasi-independent set

By the Second Decomposition Theorem we know that there is a (128ρ)-quasi-independent set P of size $\frac{n}{4}$ in $V(G)$, where $0<\varepsilon \ll d \ll \rho$. More precisely, if $n=4t+r$ and $r=0$ then $|P|=t$, otherwise $P=t+1$. Let us fix ρ to have $128\rho=10^{-100}$ (this number is small enough for our purposes, and convenient to calculate with it). It is useful to perform the following procedure: if there is a vertex $v\in V(G)-P$ and a vertex $u\in P$ such that u has more neighbors in P than v, we interchange them. There are two kind of exceptional vertices which may cause problems. First, the vertices of $Exc_1\subset P$, which do not have enough neighbors in $V(G)-P$, more precisely, $u\in Exc_1$, if $\deg_G(u,V(G)-P)\le(1-10^{-50})\frac{3n}{4}$. Second, $v\in Exc_2\subset V(G)-P$, if $\deg_G(v,P)\le(1-10^{-50})\frac{n}{4}$. An easy calculation shows that neither $|Exc_1|$, nor $|Exc_2|$ can be bigger than $10^{-50}n$.

From the fact, that $\delta_G\ge\frac{3n-1}{4}$ it is easy to see, that the minimum degree in $V(G)-P$ is always big enough to provide a triangle cover in it by the Hajnal–Szemerédi theorem. Depending on r we may have at most two vertices left out from this triangle cover. We will take care of them later.

Our strategy is to embed the 4-cliques of $V(G)-P$ in such a way, that for almost all the cliques one vertex is embedded into P and the three others into $V(G)-P$, $I(\widetilde{H})$ is embedded into P, and $\widetilde{H}-I(\widetilde{H})$ into the triangles of $V(G)-P$. Our procedure will guarantee that either we can do this, or we can find another quasi-independent set of size $\frac{n}{4}$, this time in $V(G)-P$.

We start by taking care of the exceptional vertices. Let us consider a $v\in Exc_2$, and first suppose that v is the only exceptional vertex in the triangle $\tau=\{v,v',v''\}$ from the triangle cover of $V(G)-P$. If we cannot extend it to a 4-clique with a P-vertex, then v has at most $2\cdot 10^{-50}n$ neighbors in P. Hence, it has almost full degree to $V(G)-P$. Note, that v' and v'' has at least $\frac{n}{4}$ common neighbors in $V(G)-P$, almost all of them are the neighbors of v. If $u\in P$ is adjacent to v, then u also has at most $2\cdot 10^{-50}n$ neighbors in P, otherwise we would have interchanged them in the beginning. It is important to observe that if $S\subset V(G)-P$, $|S|<\frac{n}{4}$ then by the minimum degree condition and the interchanging operation we performed between the vertices of P and $V(G)-P$, $|P\cap N_G(S)|\ge|S|$, no matter what is the value of r. Hence, there are enough neighbors for Exc_2 in P. (Here we need the full strength of the minimum degree condition.) From these facts it follows that we can find a "switching triangle" τ' for τ in such a way that one vertex of τ' changes place with v, in the new τ' v has a common P-neighbor with the other vertices, and there is no exceptional vertex in the new τ.

Next we assume that $v,w \in Exc_2$ are in the same triangle $\tau = \{v,w,v'\}$. If there is no common neighbor for τ in P, then v and w can be separated by placing them into disjoint triangles of the cover in such a way, that both triangles contain only one exceptional vertex. This is a simple consequence of the fact that vertices in Exc_2 have almost full degree to $V(G)-P$.

The last case to consider is when τ is a triangle inside Exc_2, i.e., $\tau = \{v,w,u\}$, where $v,w,u \in Exc_2$. Assuming that there is no common neighbor in P for the vertices of τ one can find a "switching triangle" for τ such that no triangle in the cover of $V(G)-P$ contains more than two vertices from Exc_2. Again, this can be shown by a simple degree counting argument.

Thus, our strategy with the vertices in Exc_2, is first to find distinct triangles for them, and then if necessary to place them into such triangles, which have common neighbor in P.

Let us continue the embedding of H with the "noise": $\widetilde{H}$. What we do is the following: We decompose $\widetilde{H}$ into a maximum independent set $I(\widetilde{H})$ and the rest, $C(\widetilde{H})$. Embed the vertices of $C(\widetilde{H})$ into the triangles of $V(G)-P$, using the least possible triangles and not touching the triangles which we used for the 4-cliques for the exceptional vertices.

Then find P-vertices for embedding $I(\widetilde{H})$, again avoiding those vertices which are used for cliques containing the exceptional vertices. That this is doable can be checked by using the König–Hall conditions.

It is possible, that $\widetilde{H}$ is not balanced, i.e., $|I(\widetilde{H})| > 3|C(\widetilde{H})|$. To make the leftover balanced, we actually put the necessary number of vertices to P, taking into account those vertices we put aside in the beginning because of the value of r. Doing so we do not produce any new Exc_2-vertices, but possibly new Exc_1-vertices. The number of these new exceptional vertices is very-very small, less than $|V(\widetilde{H})|$. Note, that the number of vertices we have not put in a 4-clique so far or used in embedding $\widetilde{H}$ is divisible by four. Hence, after making the leftover balanced, we still have a triangle cover in the leftover of $V(G)-P$. We will denote the resulting two sets by $V(G)-P$ and P, respectively.

Let us consider now the vertices of Exc_1. Assume, that $v \in Exc_1$ cannot be matched to a triangle, i.e., v may have degree 3 to some triangles, but we cannot match all Exc_1 to one-one triangle in $V(G)-P$. What we want is to find 4-cliques containing the exceptional vertices of Exc_1. Note, that if for all $v \in Exc_1$ $\deg_G(v, V(G)-P) \geq 10^{-50}n + \frac{n}{2}$, then we can match the Exc_1-

vertices to triangles. This implies that when this matching is not possible, Exc_2 is empty (recall the interchange operation). Also, v has exactly two edges to most of the triangles in the cover: $\frac{n}{2} \leq \deg_G(v, V(G)-P) < 10^{-50}n + \frac{n}{2}$ implies that v has exactly two neighbors in at least $\frac{n}{4} - 3\cdot 10^{-50}n > \frac{n}{4} - 10^{-49}n$ triangles. We are going to consider only these triangles now, denote the set of them by T. In the ith triangle of T call the vertex a_i which is not connected to v, and denote by b_i, c_i those, which are adjacent to v. If some a_i has degree 3 to another triangle in T, then we can interchange a_i with v. This way we still have the triangle cover in $V(G)-P$, and we have one more 4-clique with one vertex in P. Let us now suppose, that this is not possible. We can conclude from it that no a_i has degree 3 to more than $3\cdot 10^{-50}n$ triangles in the cover. Denote the set of a_is by A ($|A| \geq \frac{n}{4} - 10^{-49}n$), and suppose that some a_j has $10^{-48}n$ neighbors in A. Let $a_k \in A$ be one such neighbor. We know that a_k is connected to b_j or c_j, but not to both of them, say, to b_j. If c_j is adjacent to b_k and c_k, then we find the following: $\{a_j, b_j, a_k\}$ constitutes a triangle, and $\{b_k, c_k, c_j\}$ is such a triangle, which is connected to v. If this is not possible, c_j has just one neighbor in the k-th triangle, then c_j "loses" one edge. Because the average is at least two edges into one triangle, either we can find the above configuration, or c_j (or b_j) is connected to at least $10^{-48}n$ triangles in the cover with full degree. But in this case we can find two 4-cliques and one triangle in the following way: b_k and c_k has a lot of common neighbors in P, and v is connected to most of them, because it has small degree into $V(G)-P$. These four vertices give one 4-clique. The other 4-clique is composed by c_j and the vertices of the triangle to which it is connected to. Finally, $\{a_j, b_j, a_k\}$ is again a triangle. In this procedure we used 3 vertices from P and 9 from $V(G)-P$, hence, we could retain the balance. We can conclude that either for all the vertices in Exc_1 we can do the above procedure, or we find a quasi-independent set of size at least $\frac{n}{4} - 10^{-49}n$ with maximum degree at most $10^{-48}n$.

All the other vertices in P have big degrees to $V(G)-P$, and $V(G)-P$ vertices have big degrees to P-vertices. It is easy to see that the triangles of $V(G)-P$ can be matched to the vertices of P. This can be shown by checking the König–Hall conditions; thus, we can find a 4-clique cover in the leftover. The embedding can be finished.

7.2. Case 2: two quasi-independent sets

Let us return to the case when we could find a big quasi-independent set in $V(G)-P$. Call this quasi-independent set Q'. By putting at most $10^{-49}n$ new vertices to it from $V(G)-(P\cup Q')$, we get a quasi-independent set Q

of size t or $t+1$: if $r=0$, then $|P|=|Q|=t$, otherwise $|P|=t$ and $|Q|=t+1$. Note that we may have to put a vertex from P to Q, in such a case choose an Exc_1-vertex (we know that $Exc_1 \neq \emptyset$).

We will divide Exc_1 into two sets. The first one we will still denote by Exc_1, this set will contain the P-vertices having less than $(1-10^{-45})\frac{n}{2}$ neighbors in $V(G)-(P \cup Q)$. The complementer of this set, Exc_1' will contain those P-vertices having less than $(1-10^{-45})\frac{n}{4}$ Q-neighbors.

It is important to observe that all vertices in Q have big degrees to P, because Exc_2 is empty. The Q-vertices can have low degrees to $V(G)-(P \cup Q)$, but any vertex in G has enough neighbors in $V(G)-(P \cup Q)$ to provide a perfect matching in it (in fact one vertex can be left out, which we put aside to be dealt with at the end of the embedding). Like in the previous case, we interchange a Q-vertex u with a $V(G)-(P \cup Q)$-vertex v if u has more neighbors in Q than v. The subset of the vertices of Q having less than $(1-10^{-45})\frac{n}{2}$ neighbors in $V(G)-(P \cup Q)$ will be denoted by Exc_3. Similarly, we denote the subset of vertices in $V(G)-(P \cup Q)$ having less than $(1-10^{-45})\frac{n}{4}$ Q-neighbors by Exc_4.

By counting the number of edges going in between P and $V(G)-(P \cup Q)$ we can give a (rude) upper bound on $|Exc_4|$: it is smaller than $10^{-40}n$. As with Exc_2, one can see that by a possible rearranging of the edges of the perfect matching, all the Exc_4-vertices have a common neighbor in Q with their matched pair. This follows from their degrees to P, because in every case $|P|=t$ while $|Q|=t$ or $t+1$ depending on r. Now we have triangles: one such triangle contains one Exc_4-vertex, another vertex from $V(G)-(P \cup Q)$, and a Q-vertex. Observe that a similar procedure can be performed for the Exc_1'-vertices of P, too. For performing the above operations, we needed the full strength of the minimum degree condition on G. For the rest of H we do not need the tight minimum degree condition.

It is easy to extend these triangles into 4-cliques: every vertex has almost full degree to P.

Now we turn our attention to embedding $\widetilde{H}$. Again, we decompose it into two parts, $I(\widetilde{H})$ and $C(\widetilde{H})$. First, we take care of the vertices of $C(\widetilde{H})$. We embed one vertex of a cycle (or path) to Q, and then the next two vertices to an edge of the perfect matching, not touching those vertices, which are in a 4-clique with some exceptional vertex. The degrees are big enough, so this can be done easily, and at most one vertex can be left out from a matching edge. Then we just choose their neighbors from P, again not touching any 4-clique with some exceptional vertex. As previously happened, after this we may arrive to an unbalanced situation. Also, we may have an extra vertex depending on the parity of $|V(G)-(P \cup Q)|$. We put the necessary number of vertices to P and maybe one to Q. Notice that this way we do not produce any Exc_4-vertex, or Exc_1'-vertex (we do not put Exc_4-vertex to P). Only Exc_1 or Exc_3 can increase. Also, there still will be a perfect matching in $V(G)-(P\cup Q)$.

Next, we will take care of the Exc_3-vertices. For each of them we want to find one-one edge in the perfect matching to which it is connected to.

By counting the degrees as we did previously, one can conclude that if there are not enough edges in $V(G)-(P\cup Q)$ connected to an Exc_3-vertex, then every vertex in Exc_4 has a large neighborhood in Q. This is because in this case at least one $v\in Exc_3$ has less than $\frac{n}{4}+10^{-40}n$ neighbors. Out of the $\frac{n}{4}$ matching edges, v has degree two to at most $10^{-40}n$, hence, it has degree one to at most $10^{-40}n$ matching edges, implying that to at least $\frac{n}{4}-10^{-39}n$ edges of the matching it has degree one. Consider now these edges, and denote the vertex of the ith edge not connected to v by a_i, and the other by b_i. Let $\{a_i,b_i\}$ and $\{a_k,b_k\}$ be two edges. If a_i is adjacent to a_k, then a $\{b_i,a_k\}$ edge would imply that we can interchange v and a_k, v is in a matching edge with b_k, and a_k is connected to the matching edge $\{a_i,b_i\}$. If $\{b_i,b_k\}$ is an edge, then we have the triangle $\{v,b_i,b_k\}$ and the matching edge $\{a_i,a_k\}$. Suppose that b_i is non-adjacent to b_k and a_k. Observing that b_i has at least $\frac{n}{4}$ neighbors inside, and this way it loses potential neighbors, we conclude that this can happen at most $10^{-38}n$ times. Hence, if even with this kind of procedure we cannot handle v, then no a_i can have more than $10^{-38}n$ other a_k neighbors: the set of a_is is a quasi-independent set of size at least $\frac{n}{4}-10^{-39}n$ in $V(G)-(P\cup Q)$, and with maximum degree $10^{-38}n$.

On the other hand, if we can find one-one edge to the Exc_3-vertices, then we can easily extend these triangles into 4-cliques, because every vertex has almost full degree to P.

Still, we haven't considered the set Exc_1. For a $v\in Exc_1$ we are looking for a neighbor $w\in Q$ and an edge $\{u_1,u_2\}$ from the matching of $V(G)-(P\cup Q)$ such that these four vertices constitute a 4-clique.

Note that v has almost full degree to Q, and it is possible that it is not connected to any matching edge in $V(G)-(P\cup Q)$ with two edges. Like in the case of Exc_3 we get that to at least $\frac{n}{4}-10^{-39}n$ matching edges v has degree one. Out of those edges consider the ones, for which none of the endpoints are from Exc_4. This way we are considering at least $\frac{n}{4}-10^{-38}n$ matching edges. Using our notation above, we get that if some a_i is connected to another a_k, then if b_i is adjacent to a_k, then we can interchange v with b_i. This way v will be put to $V(G)-(P\cup Q)$, in the matching edge $\{v,b_k\}$, and $\{a_k,a_i,b_i\}$ is a triangle. Both this edge and triangle are easy to extend to a 4-clique. Otherwise, if b_i is adjacent to b_k, then we have the 4-clique we wanted given by the vertices v, b_i, b_k and a Q-vertex connected to them. The matching edge will be the $\{a_i, a_k\}$-edge. Like above, we can conclude that if we cannot handle v, then the set of a_is is a quasi-independent set of size at least $\frac{n}{4} - 10^{-39}n$ and maximum degree $10^{-38}n$.

The leftover are non-extremal vertices. These all have very small degrees in their set (P, Q or $V(G)-(P\cup Q)$). We will match an edge of the matching in $V(G)-(P\cup Q)$ by a P-vertex and a Q-vertex, such that these four vertices constitute a 4-clique. That this is doable can be shown by checking the König–Hall conditions.

7.3. Case 3: quasi-four-partite graph

To complete our proof we have to show the existence of an embedding, when G is a quasi-four-partite graph. Recall that in this case we cannot find 4-cliques for all vertices of Exc_1 and Exc_3.

Let us denote the last two quasi-independent sets by R and S. The fact that both are quasi-independent is the consequence of R being quasi-independent, and that we prepared G by deleting edges which were not necessary for satisfying the minimum degree condition. Now all of our quasi-independent sets are of size t or $t+1$.

Note that $V(\widetilde{H})$ has an equipartition to 4 independent sets by the Hajnal–Szemerédi theorem. We distribute these vertices according to the sizes of P, Q, R and S. Thus, after embedding $\widetilde{H}$ the remaining sets will be of the same size.

The embedding is now done as follows. First we find 4-cliques for the exceptional vertices. All the exceptional sets in the quasi-independent sets will be small. We say v is exceptional if it has more than $10^{-20}n$ neighbors in its quasi-independent set. Then the exceptional sets will be smaller than $10^{-20}n$. Let us consider e.g., what we do with a $v \in Exc_1$. If say, v has small degree to Q than v has a lot of neighbors in P, R and S. Find one-one neighbor in these sets, which together with v constitute a 4-clique. Then, because v had small degree to Q, we can find two Q-vertices, which are connected, and one-one from R and S such that these four vertices constitute a 4-clique. The two vertices from Q should have good degrees to R and S. This is not a significant restriction since almost all of them have good degrees, and because with most of them v is not connected, they have an extra edge inside. For all the exceptional vertices one can find a 4-clique applying a similar procedure. Then one can embed $\widetilde{H}$ easily: use the equipartition of $V(\widetilde{H})$ provided by the Hajnal–Szemerédi theorem, and put the vertices of the different partitions into different quasi-independent sets.

One has to be careful when choosing the quasi-independent set: it is picked according to the sizes of the quasi-independent sets and the partitions of $V(\widetilde{H})$.

Then the matching for the leftover can be done in the following order. First, match the rest of S with the rest of R. Then match these edges with the vertices of Q. The last step is to match these triangles with the vertices of P. The existence of all these matchings can be verified easily, on the same way we did it previously. ∎

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